Blow up of Solution for the Generalized Boussinesq Equation with Damping Term

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We consider the blow up of solution to the initial boundary value problem for the generalized Boussinesq equation with damping term. Under some assumptions we prove that the solution with negative initial energy blows up in finite time.

Key words: Damped Boussinesq Equation; Generalized Boussinesq Equation; Blow up of Solution; Initial Boundary Value Problem.

1. Introduction

In this paper, we study the blow up of solution of the following initial boundary value problem for the generalized Boussinesq equation with damping term:

$$u_{tt} - u_{xx} + \delta u_{xxxx} - \lambda u_{xxtt} - r u_{xxt} = \beta (u^2)_{xx} - p^2 u, \quad x \in \mathrm{R},\ t > 0, \tag{1}$$

$$u(-\infty,t) = u(+\infty,t) = u_{xx}(-\infty,t) = u_{xx}(+\infty,t) = 0, \quad t \geq 0, \tag{2}$$

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \quad x \in \mathrm{R}, \tag{3}$$

where δ, λ, $r \geq 0$ are constants, $p \neq 0$ and $\beta \in \mathrm{R}$.

Scott Russell's study [1] of solitary water waves motivated the development of nonlinear partial differential equations for the modeling wave phenomena in fluids, plasmas, elastic bodies, etc. Especially when $p = r = \lambda = 0$ and $\beta = 1$ equation (1) becomes the Boussinesq equation

$$u_{tt} - u_{xx} + \delta u_{xxxx} = (u^2)_{xx},$$

which is an important model that approximately describes the propagation of long waves on shallow water like the other Boussinesq equations (with u_{xxtt} instead of u_{xxxx}). This equation was first deduced by Boussinesq [2]. In the case $\delta > 0$ this equation is linearly stable and governs small nonlinear transverse oscillations of an elastic beam (see [3] and references therein). It is called the "good" Boussinesq equation, while the equation with $\delta < 0$ received the name "bad" Boussinesq equation since it possesses linear instability.

There is a considerable mathematical interest in the Boussinesq equations which have been studied from various aspects (see [4 – 7] and references therein). A great deal of efforts has been made to establish sufficient conditions for the nonexistence of global solutions to various associated boundary value problems [6, 8].

Levine and Sleeman [8] studied the global nonexistence of solutions for the equation

$$u_{tt} - u_{xx} - 3u_{xxxx} + 12(u^2)_{xx} = 0$$

with periodic boundary conditions. Turitsyn [6] proved the blow up in the Boussinesq equations

$$u_{tt} - u_{xx} + u_{xxxx} + (u^2)_{xx} = 0$$

and

$$u_{tt} - u_{xx} - u_{xxtt} + (u^2)_{xx} = 0$$

for the case of periodic boundary conditions and obtained exact sufficient criteria of the collapse dynamics. The generalization of the Boussinesq equation

$$u_{tt} - u_{xx} + \alpha u_{xxxx} + f(u)_{xx} = 0$$

was studied in numerous papers [9 – 17].

Liu [13, 14] studied the instability of solitary waves for the generalized Boussinesq type equation

$$u_{tt} - u_{xx} + (f(u) + u_{xx})_{xx} = 0,$$

and established some blow up result for a nonlinear Pochhammer-Chree equation

$$u_{tt} - u_{xxtt} - f(u)_{xx} = 0.$$

Zhijian [16], Yang and Wang [17] studied the blow up of solutions to the initial boundary value problems for the Boussinesq equations

$$u_{tt} - u_{xx} - bu_{xxxx} = \sigma(u)_{xx}$$

and

$$u_{tt} - u_{xx} - u_{xxtt} = \sigma(u)_{xx}.$$

Godefroy [9] showed the blow up of the solutions for the following problem:

$$u_{tt} = f(u)_{xx} + u_{xxtt}, \quad x \in \mathrm{R},\ t \geq 0,$$
$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x),$$
$$u(+\infty,t) = u(-\infty,t) = 0,$$

where $f:\ R \to R\, C^\infty,\ f(0) = 0$.

In the Boussinesq equations, the effects of small nonlinearity and dispersion is taken into consideration, but in many real situations damping effects are compared in strength to the nonlinear and dispersive one. Therefore the damped Boussinesq equation is considered as well:

$$u_{tt} - 2bu_{txx} = -\alpha u_{xxxx} + u_{xx} + \beta(u^2)_{xx},$$

where u_{txx} is the damping term and $\alpha, b = \text{const.} > 0$, $\beta = \text{const.} \in \mathrm{R}$ (see [3–5, 7, 18] and references therein).

Varlamov [3, 7] investigated the long-time behavior of solutions to initial value, spatially periodic, and initial boundary value problems for the damped Boussinesq equation in two space dimensions. Polat et al. [15] established the blow up of the solutions for the damped Boussinesq equation.

Lai and Wu [4], Lai et al. [5] investigated, respectively, the global solution of the following generalized damped Boussinesq equations:

$$u_{tt} - au_{ttxx} - 2bu_{txx} = -cu_{xxxx} + u_{xx} - p^2u + \beta(u^2)_{xx}$$

and

$$u_{tt} - au_{ttxx} - 2bu_{txx} = -cu_{xxxx} + u_{xx} + \beta(u^2)_{xx}.$$

In this paper, we establish a blow up result for a solution with negative initial energy of initial boundary value problem for the generalized Boussinesq equation with damping term (1)–(3).

2. Existence and Uniqueness for the Solution

Theorem 1. Suppose that $u_0(x) \in H^{s+1}$ and $u_1(x) \in H^s$ with $s > \frac{1}{2}$. If $\lambda > 0, r > 0$ and $\lambda + \delta > \frac{r^2}{4}$ then there exists a unique solution $u(x,t) \in C([0,T], H^{s+1}) \cap C^1([0,T], H^s)$ to the problem defined by (1)–(3).

Proof: For this we refer to Lai et al. [5].

3. Blow up of Solution

We prove first the following lemma.

Lemma 1. If there exist functions $w_0(x) \in H^{s+2}$ and $v_0(x) \in H^{s+1}$ such that the initial values $u_0(x)$ and $u_t(x,0)$ satisfy the relations

$$u(x,0) = (w_0(x))_x, \quad u_t(x,0) = (v_0(x))_x,$$

then for all $t \in T$ the solution $u(x,t)$ of (1)–(3) satisfies $u(x,t) = (w(x,t))_x$, with a corresponding evolution of $w(x,t)$, $v(x,t)$ satisfying the system

$$\begin{aligned} &w_t(x,t) = v(x,t),\\ &v_t - rv_{xx} - \lambda v_{xxt}\\ &\quad = w_{xx} - \delta w_{xxxx} + \beta({w_x}^2)_x + p^2 w. \end{aligned} \tag{4}$$

Proof. Writing (1) in the form

$$\begin{aligned} &u_t = z_x,\\ &z_t - rz_{xx} - \lambda z_{xxt}\\ &\quad = (u - \delta u_{xx} + \beta u^2)_x - \int^x p^2 u \,\mathrm{d}\eta \end{aligned} \tag{5}$$

and using (5) we obtain $u(x,t) = u(x,0) + \int_0^t z_x(x,s)\mathrm{d}s$. The term $u(x,0)$ is an x-derivative by hypothesis and $\int_0^t z_x(x,s)\mathrm{d}s$ is an x-derivative. Therefore there exists a $w(x,t)$ such that $u(x,t) = (w(x,t))_x$, which gives, from (1),

$$\begin{aligned} &u = w_x,\\ &w_{tt} - w_{xx} + \delta w_{xxxx} - \lambda w_{xxtt} - rw_{xxt}\\ &\quad = \beta({w_x}^2)_x - p^2 w. \end{aligned} \tag{6}$$

Hence we easily obtain the system (4).

By the Lemma, the initial boundary value problem (1)–(3) corresponds the following problem:

$$\begin{aligned} &w_{tt} - w_{xx} + \delta w_{xxxx} - \lambda w_{xxtt} - rw_{xxt}\\ &\quad = \beta({w_x}^2)_x - p^2 w, \quad x \in \mathrm{R},\ t > 0, \end{aligned} \tag{7}$$

$$w_x(-\infty,t) = w_x(+\infty,t) = w_{xxx}(-\infty,t) = w_{xxx}(+\infty,t) = 0, \quad t>0, \tag{8}$$

$$w(x,0) = w_0(x),\ w_t(x,0) = w_1(x), \quad x \in \mathrm{R}. \tag{9}$$

Theorem 2. Suppose the initial values $u(x,0)$ and $u_t(x,0)$ are choosen such that they satisfy the following assumptions:

(i) $u(x,0) = (w_0(x))_x$, $u_t(x,0) = (v_0(x))_x$ for some $w_0 \in H^{s+2}$ and $v_0 \in H^{s+1}$, $s > \frac{1}{2}$;

(ii) $E(0) = \frac{1}{2}\|v(0)\|_2^2 + \frac{p^2}{2}\|w(0)\|_2^2 + \frac{1}{2}\|w_x(0)\|_2^2 + \frac{\delta}{2}\|w_{xx}(0)\|_2^2 + \frac{\lambda}{2}\|v_x(0)\|_2^2 + \frac{\beta}{3}\int_{-\infty}^{+\infty} {w_{0x}}^3 \mathrm{d}x < 0.$

Then the solution $u(x,t)$ of problem (1)–(3) blows up in finite time.

Proof. In order to prove Theorem 2, we will use the following lemma.

Lemma 2. Suppose that a positive, twice differentiable function $F(t)$ satisfies on $t \geq 0$ the inequality

$$F''(t)F(t) - (1+\upsilon)F'^2(t) \geq 0,$$

where $\upsilon > 0$ is a constant. If $F(0) > 0$ and $F'(0) > 0$, then there exists a positive constant $\tilde{T} \leq \frac{F(0)}{\alpha F'(0)}$ such that $F(t) \to \infty$ as $t \to \tilde{T}^-$.

Taking L^2 inner product of equation (7) with w_t and integrating the new equation over $(0,t)$, we obtain

$$\int_{-\infty}^{+\infty} w_t w_{tt} \mathrm{d}x - \int_{-\infty}^{+\infty} w_t w_{xx} \mathrm{d}x + \delta \int_{-\infty}^{+\infty} w_t w_{xxxx} \mathrm{d}x - \lambda \int_{-\infty}^{+\infty} w_t w_{xxtt} \mathrm{d}x - r \int_{-\infty}^{+\infty} w_t w_{xxt} \mathrm{d}x$$
$$= \beta \int_{-\infty}^{+\infty} w_t ({w_x}^2)_x \mathrm{d}x - p^2 \int_{-\infty}^{+\infty} w_t w \mathrm{d}x,$$
$$E'(t) + r\|w_{xt}(t)\|_2^2 = 0 \Rightarrow$$
$$E(t) + r\int_0^t \|w_{xt}(s)\|_2^2 \mathrm{d}s = E(0) < 0, \quad t \geq 0, \tag{10}$$

where

$$E(t) = \frac{1}{2}\|w_t(t)\|_2^2 + \frac{p^2}{2}\|w(t)\|_2^2 + \frac{1}{2}\|w_x(t)\|_2^2 + \frac{\delta}{2}\|w_{xx}(t)\|_2^2 + \frac{\lambda}{2}\|w_{xt}(t)\|_2^2 + \frac{\beta}{3}\int_{-\infty}^{+\infty} {w_x}^3 \mathrm{d}x, \quad t \geq 0. \tag{11}$$

Let

$$F(t) = \|w(t)\|_2^2 + \lambda\|w_x(t)\|_2^2 + r\int_0^t \|w_x(s)\|_2^2 \mathrm{d}s + (T^* - t)\|w_{0x}\|_2^2 + \beta_0(t+\tau)^2, \quad t \in [0, T^*], \tag{12}$$

where T^*, β_0 and τ are positive constants to be specified later. Suppose that the maximal time of existence of the solution for problem (1)–(3) is infinite. A contradiction will be obtained by Lemma 2. Obviously we have

$$F'(t) = 2\left[(w,w_t) + \lambda(w_x,w_{xt}) + r\int_0^t (w_x,w_{xt})\mathrm{d}s + \beta_0(t+\tau)\right]. \tag{13}$$

Using the Schwarz inequality and the inequality $(a_1b_1 + \ldots + a_nb_n)^2 \leq (a_1^2 + \ldots + a_n^2)(b_1^2 + \ldots + b_n^2)$, where $a_i, b_i \geq 0$ $(i = 1, \ldots, n)$, we have

$$F'^2(t) \leq 4\left[\|w(t)\|_2^2 + \lambda\|w_x(t)\|_2^2 + r\int_0^t \|w_x(s)\|_2^2 \mathrm{d}s + \beta_0(t+\tau)^2\right]\left[\|w_t(t)\|_2^2 + \lambda\|w_{xt}(t)\|_2^2 + r\int_0^t \|w_{xt}(s)\|_2^2 \mathrm{d}s + \beta_0\right]$$
$$= 4\left[F(t) - (T^* - t)\|w_{0x}\|_2^2\right]\left[\|w_t(t)\|_2^2 + \lambda\|w_{xt}(t)\|_2^2 + r\int_0^t |w_{xt}(s)\|_2^2 \mathrm{d}s + \beta_0\right]$$
$$\leq 4F(t)\left(\|w_t(t)\|_2^2 + \lambda\|w_{xt}(t)\|_2^2 + r\int_0^t \|w_{xt}(s)\|_2^2 \mathrm{d}s + \beta_0\right), \tag{14}$$

$$F''(t) = 2\left[\|w_t(t)\|_2^2 + (w,w_{tt}) + \lambda\|w_{xt}(t)\|_2^2 + \lambda(w_x,w_{xtt}) + r(w_x,w_{xt}) + \beta_0\right]$$
$$= 2\left(\|w_t(t)\|_2^2 + \lambda\|w_{xt}(t)\|_2^2 - p^2\|w(t)\|_2^2 - \|w_x(t)\|_2^2 - \delta\|w_{xx}(t)\|_2^2 - \beta\int_{-\infty}^{+\infty} {w_x}^3 \mathrm{d}x + \beta_0\right)$$
$$= 2\left(\frac{5}{2}\|w_t(t)\|_2^2 + \frac{5}{2}\lambda\|w_{xt}(t)\|_2^2 + \frac{p^2}{2}\|w(t)\|_2^2 + \frac{1}{2}\|w_x(t)\|_2^2 + \frac{\delta}{2}\|w_{xx}(t)\|_2^2 + 3r\int_0^t \|w_{xt}(s)\|_2^2 \mathrm{d}s - 3E(0) + \beta_0\right), \tag{15}$$

where (10) has been used. From (13) – (15) we obtain

$$F(t)F''(t) - \left(1+\frac{\alpha}{4}\right)F'^2(t) \geq F(t)F''(t) - (4+\alpha)F(t)\left(\|w_t(t)\|_2^2 + \lambda\|w_{xt}(t)\|_2^2 + r\int_0^t \|w_{xt}(s)\|_2^2 \mathrm{d}s + \beta_0\right)$$
$$= F(t)\Big[(1-\alpha)\|w_t(t)\|_2^2 + (1-\alpha)\lambda\|w_{xt}(t)\|_2^2 + p^2\|w(t)\|_2^2 + \|w_x(t)\|_2^2 \qquad (16)$$
$$+ \delta\|w_{xx}(t)\|_2^2 + (2-\alpha)r\int_0^t \|w_{xt}(s)\|_2^2 \mathrm{d}s - 6E(0) - (2+\alpha)\beta_0\Big].$$

Taking $0<\alpha\leq 1$, $\beta_0 = -\frac{6}{2+\alpha}E(0) > 0$, then from the inequality (16), we get

$$F(t)F''(t) - \left(1+\frac{\alpha}{4}\right)F'^2(t) \geq 0, \quad t\in[0,T^*]. \quad (17)$$

We may now choose τ and T^* such that

$$(w_0,w_1)+\lambda(w_{0x},w_{1x})+\beta_0\tau > 0,$$
$$\alpha\left[(w_0,w_1)+\lambda(w_{0x},w_{1x})+\beta_0\tau\right] > 2\|w_{0x}\|_2^2, \quad (18)$$
$$T^* \geq \frac{\|w_0\|_2^2+\lambda\|w_{0x}\|_2^2+\beta_0\tau^2}{\alpha[(w_0,w_1)+\lambda(w_{0x},w_{1x})+\beta_0\tau]-2\|w_{0x}\|_2^2}.$$

Then

$$F'(0) = 2[(w_0,w_1)+\lambda(w_{0x},w_{1x})+\beta_0\tau] > 0,$$
$$\tilde{T}^- = \frac{4F(0)}{\alpha F'(0)} = \frac{2(\|w_0\|_2^2+\lambda\|w_{0x}\|_2^2+T^*\|w_{0x}\|_2^2+\beta_0\tau^2)}{\alpha[(w_0,w_1)+\lambda(w_{0x},w_{1x})+\beta_0\tau]} \leq T^*. \quad (19)$$

From Lemma 2 we know that $F(t)$ becomes infinite at a time T_1 at most equal to

$$\tilde{T}^- = \frac{4F(0)}{\alpha F'(0)} \leq \infty.$$

And therefore we have a contradiction with the fact that the maximal time of existence is infinite. Hence there is a blow up in finite time, and the maximal time of existence is finite. So

$$\|w(t)\|_2^2 + \|w_x(t)\|_2^2 + \int_0^t \|w_x(s)\|_2^2 \mathrm{d}s \to \infty \text{ as } t\to \tilde{T}^-.$$

This completes the proof.

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